

Problem Set 7 Solutions

Igor Rapinchuk

1. Suppose $|G| = p^n$. Let S_1, \dots, S_k be the orbits of G on S . Then

$$|S| = |S_1| + \dots + |S_k|$$

Suppose that G does not have a fixed point. Then the size of every orbit is > 1 . On the other hand, $|S_i| = [G : G_{x_i}]$ where $x_i \in S_i$. It follows that $|S_i| = p^{n_i}$ for some $n_i > 0$, in particular, $|S_i|$ is divisible by p . But then $|S|$ is also divisible by p . A contradiction.

2. Let G be the dihedral group of order $2n$,

$$G = \{1, x, \dots, x^{n-1}, y, xy, \dots, x^{n-1}y\}$$

where $x^n = 1$, $y^2 = 1$, and $yx y^{-1} = x^{-1}$. We have

$$x^i x^j (x^i)^{-1} = x^j \quad \text{and} \quad (x^i y) x^j (x^i y)^{-1} = x^{-j}$$

So, the conjugacy class of x^j is $\{x^j, x^{-j}\}$. Next,

$$\begin{aligned} x^i (x^j y) (x^i)^{-1} &= x^{2i} (x^j y) \quad \text{and} \quad (x^i y) (x^j y) (x^i y)^{-1} = x^i y x^j y y^{-1} x^{-i} = \\ &= x^{2i-j} y = x^{2(i-j)} (x^j y) \end{aligned}$$

So, the conjugacy class of $x^j y$ is $\{x^{2i} (x^j y) \mid i = 0, \dots, n-1\}$.

Case 1. n is odd, $n = 2m+1$. Then $x^j \neq x^{-j}$ for any $j = 1, \dots, n-1$. So, the powers of x split into one 1-element conjugacy class $\{1\}$ and m 2-element conjugacy classes $\{x^j, x^{-j}\}$, $j = 1, \dots, m$. Since n is odd, any power of x can be written in the form x^{2i} , so all elements of the form $x^j y$ form a single conjugacy class. So, the class equation for G looks as follows:

$$2n = 1 + \underbrace{2 + \dots + 2}_m + n$$

Case 2. n is even, $n = 2m$. Then $x^j = x^{-j}$ iff $j = 0$ or m , so there are two 1-element conjugacy classes $\{1\}$ and $\{x^m\}$. The conjugacy class of any other power of x contains 2 elements; these classes are $\{x^j, x^{-j}\}$, where $j = 1, \dots, m-1$. Since n is even, there are m elements of the form x^{2i} . So, the conjugacy class of an element of the form $x^j y$ contains m elements, and therefore there are 2 such classes. So, the class equation for G looks as follows:

$$2n = 1 + 1 + \underbrace{2 + \dots + 2}_{m-1} + m + m$$

3. Suppose G/Z is generated by a coset gZ . This means that any coset xZ is of the form $g^i Z$ for some integer i . It follows that any $x \in G$ can be written in the form $g^i z$ for $z \in Z$. Now, let us take two arbitrary $a, b \in G$ and write them in the form $a = g^i z_1$ and $b = g^j z_2$. Then

$$ab = (g^i z_1)(g^j z_2) = g^{i+j} z_1 z_2$$

and

$$ba = (g^j z_2)(g^i z_1) = g^{i+j} z_2 z_1$$

because z_1 and z_2 commute with g . Since $z_1 z_2 = z_2 z_1$, we see that $ab = ba$, so G is abelian.

4. Counter-example: Let $G = S_3$, $H = \{1, \tau_1\}$. Since $[G : H] = 3$ and H is not normal, the normalizer $N(H)$ coincides with H . So, $N(H)$ is not normal in G .

5. By definition, $N(H) = \{g \in G \mid gHg^{-1} = H\}$. If H is normal in K , for any $k \in K$ we have $kHk^{-1} = H$. This means that $k \in N(H)$, and therefore $K \subset N(H)$. Conversely, suppose $K \subset N(H)$. Then for any $k \in N(H)$, we have $kHk^{-1} = H$, so H is normal in K .

6. The number of Sylow 5-subgroups must divide 4 and must be congruent to 1 modulo 5. So, the number is 1. The Sylow 5-subgroup is cyclic of order 5, so it contains 4 elements of order 5.

7. By First Sylow Theorem, G contains a subgroup H of order p^e . So, it is enough to show that H contains a subgroup of order p^r for every $r \leq e$. We will argue by induction on e . If $e = 1$, then H obviously has subgroups of orders 1 and p . Suppose that our claim is true for $e = k$; then we will prove that it is also true for $e = k + 1$. By Proposition 6.1.11, the center $Z(H)$ is nontrivial. Let $x \in Z(H)$ be a nonidentity element. Then the order of x is p^l , $l \geq 1$. Replacing x with $x^{p^{l-1}}$, we can assume that the order of x is p . Let K be the cyclic subgroup generated by x . Then K has order p and is normal in H because x is central in H . Consider the group homomorphism $\varphi: H \rightarrow H/K$, $\varphi(x) = xK$. The order of H/K is p^k , so the induction hypothesis applies. Then for each $r = 1, \dots, k$, the quotient group H/K has a subgroup L_r of order p^r . Set $H_r = \varphi^{-1}(L_r)$. Since φ induces a surjective homomorphism $H_r \rightarrow L_r$ with kernel K , the order of H_r is p^{r+1} . Thus, H has subgroups of order p^l for $l = 2, \dots, k + 1$. But H also has a subgroup of order p , for example, K . So, H has a subgroup of order p^r for every $r = 0, 1, \dots, e = k + 1$.

8. Suppose first that $a = 1$, and then $e > 1$. In this case, G is a p -group. By Proposition 1.11, G has nontrivial center. Then the center contains an element x of order p . Let H be the cyclic subgroup generated by x . Since x is central, we have $gx^i g^{-1} = x^i$ for any i and any $g \in G$, implying that H is a normal subgroup of order p , so it is a required nontrivial normal subgroup.

Suppose now that $a > 1$. Then the number m of Sylow p -subgroups must divide a and must be congruent to 1 modulo p . This means that $m = 1 + kp$. Since m divides a , we have $m \leq a < p$. But if $k \geq 1$, we have $m > p$, which is impossible. So, $m = 1$ which means that the Sylow p -subgroup is normal.

9. Theorem 5.5.1 lists 5 possibilities for a group G of order 12. In cases (i) and (ii), the group G is commutative, so the class equation is

$$12 = \underbrace{1 + \dots + 1}_{12}$$

If $G = A_4$, we argue as follows. The order of an element of S_4 can only be 1, 2, 3, or 4. Elements of order 4 are the cycles of length 4, which are odd permutations. So, the order of an element of order of A_4 can only be 1, 2, or 3. The elements of order 3 are the cycles of length 3, which are even permutation. So, all cycles of length 3 belong to A_4 . The even permutations of order 2 are products of two transpositions, and these are (12)(34), (13)(24), and (14)(23). It follows that A_4 contains $12 - 4 = 8$ cycles of length 3. The centralizer of a cycle of length 3 in S_4 is the cyclic subgroup generated by this cycle, so the

entire centralizer is contained in A_4 and has order 3. This means that the conjugacy class of a cycle of length 3 contains $12 : 3 = 4$ elements, so these cycles split into 2 conjugacy classes. Now, let us find the centralizers of elements of order 2. Such an element cannot commute with a cycle of length 3, because as we said earlier, the centralizer of a cycle of length 3 is the cyclic subgroup generated by this cycle. So, the centralizer of each of the elements of order 2 is the abelian subgroup

$$H = \{e, (12)(34), (13)(24), (14)(23)\}.$$

This means that the conjugacy class of an element of order 2 contains $12 : 4 = 3$ elements. Thus, these three elements form a single conjugacy class. So, the class equation for A_4 looks as follows:

$$12 = 1 + 3 + 4 + 4$$

If $G = D_{12}$, the class equation was determined in Problem 2:

$$12 = 1 + 1 + 2 + 2 + 3 + 3$$

So, it remains to consider the case where G is generated by x, y with relations $x^4 = 1$, $y^3 = 1$, $xy = y^2x$ (notice that the last relation is equivalent to $xyx^{-1} = y^{-1}$.) Let H be the cyclic group generated by x and K be the cyclic group generated by y . The relation $xyx^{-1} = y^{-1}$ implies that the normalizer of K contains x and y , and therefore coincides with G . So, K is normal. On the other hand, K is a Sylow 3-subgroup of G . So, K is a unique Sylow 3-subgroup and therefore contains all elements of order 3 in G . Thus, G has two elements of order 3, y and y^{-1} , and these elements form one conjugacy class because $xyx^{-1} = y^{-1}$. Next, H is a Sylow 2-subgroup of G . Since any element of order 2 and 4 is contained in some Sylow 2-subgroup and any two Sylow 2-subgroups are conjugate, we conclude that any element of order 2 or 4 is conjugate to x^i , $i = 1, 2, 3$. The centralizer of x^2 contains x and y (because $x^2yx^{-2} = (y^{-1})^{-1} = y$), so x^2 is a central element. It follows that x^2 is the only element of order 2 in G , so it is its own conjugacy class. Since $y^{-1}xy = y^{-2}x = yx$, the subgroup H is not normal. As $[G : H] = 3$, the normalizer $N(H)$ can only have index 1 or 3, and since H is not normal, we conclude that $[G : N(H)] = 3$. This means that G has 3 Sylow 2-subgroups. Since no two distinct Sylow 2-subgroups can have a common element of order 4, we see that G has 6 elements of order 4 (they are conjugate to x or x^{-1}). Clearly, the centralizer of x contains H , but is not equal to all of G because x is not central, so $C(x) = C(x^{-1}) = H$. So, x and x^{-1} each have 3 conjugates. Thus, G has 2 conjugacy classes of elements of order 4, each containing 3 elements. It remains to consider elements of order 6. Since G does not have elements of order 12, our previous computations show that G has $12 - (1 + 1 + 2 + 6) = 2$ elements of order 6. Since such an element is a product of an element of order 2 and an element of order 3, and the only element of order 2 is x^2 and the only elements of order 3 are y and y^{-1} , we have the following possibilities for elements of order 6: x^2y and x^2y^{-1} . Since x^2 commutes with y , these elements indeed have order 6. The equation $x(x^2y)x^{-1} = xy^{-1}$ shows that they are conjugate. So, there is one conjugacy class of elements of order 6 containing 2 elements. So, the class equation for G is

$$12 = 1 + 1 + 2 + 3 + 3 + 2$$

10. If $p = 2$ then G has order 4, hence abelian. Then it is either cyclic or the direct product of two cyclic groups of order 2, which is the dihedral group of order 4. So, suppose $p > 2$. Let K be a Sylow 2-subgroup, and H be a Sylow p -subgroup. Then $|K| = 2$ and

$|H| = p$, so both K and H are cyclic. Let y be a generator of K and x be a generator of H . The subgroup H is always normal because the number of Sylow p -subgroups must divide 2 and must be congruent to 1 modulo p , so the number is 1 and H is normal. Then $yx y^{-1} = x^a$ for some integer a . Since $y^2 = 1$, we have

$$x = y^2 x (y^{-1})^2 = y (y x y^{-1}) y^{-1} = y (x^a) y^{-1} = (x^a)^a = x^{a^2}$$

It follows that $a^2 \equiv 1 \pmod{p}$; in other words, $a^2 - 1 = (a + 1)(a - 1)$ is divisible by p . Then either $a \equiv 1 \pmod{p}$ or $a \equiv -1 \pmod{p}$. In the first case, $yx y^{-1} = x$, and therefore $yx = xy$. Then H and K commute elementwise, implying that the map $H \times K \rightarrow G$ is a group homomorphism. Since $H \cap K = \{1\}$ because $|H| = p$ (odd prime) and $|K| = 2$, we conclude that this homomorphism is injective (Proposition 2.8.6). Since $|H \times K| = 2p = |G|$, it is also surjective. Thus $G \approx H \times K$. But $H \times K$ is cyclic because the element (x, y) has order $2p$. It follows that G is also cyclic with generator xy .

Now, suppose $a \equiv -1 \pmod{p}$. Then $yx y^{-1} = x^{-1}$. Then the elements $1, x, \dots, x^{p-1}, y, xy, \dots, x^{p-1}y$ are all distinct (because x has order p and a relation $x^i y = x^j$ would imply $y = x^{j-i} \in K \cap H = \{1\}$ and therefore $y = 1$, a contradiction). Thus,

$$G = \{1, x, \dots, x^{p-1}, y, xy, \dots, x^{p-1}y\}$$

with the relations $x^p = 1$, $y^2 = 1$ and $yx y^{-1} = y^{-1}$, and this is precisely the dihedral group D_{2p} .